

A NEW PROOF OF FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE FINITENESS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let R denote a commutative Noetherian ring. Brodmann et al. in [5] defined and studied the concept of the local-global principle for annihilation of local cohomology modules at level $r \in \mathbb{N}$ for the ideals \mathfrak{a} and \mathfrak{b} of R . It was shown in [5] that this principle holds at levels 1, 2, over R and at all levels whenever $\dim R \leq 4$. The goal of this paper is to show that, if the set $\text{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite or $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$, then the local-global principle holds at all levels $r \in \mathbb{N}_0$, for all ideals $\mathfrak{a}, \mathfrak{b}$ of R and each finitely generated R -module M , where $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ denotes the first non \mathfrak{b} -cofiniteness of local cohomology module $H_{\mathfrak{a}}^i(M)$. As a consequence of this, we provide a new and short proof of the Faltings' local-global principle for finiteness dimensions. Also, several new results concerning the finiteness dimensions are given.

1. INTRODUCTION

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and \mathfrak{a} an ideal of R . For an R -module M , the i^{th} local cohomology module of M with support in $V(\mathfrak{a})$ is defined as:

$$H_{\mathfrak{a}}^i(M) = \varinjlim_{n \geq 1} \text{Ext}_R^i(R/\mathfrak{a}^n, M).$$

We refer the reader to [6] or [9] for more details about local cohomology. An important theorem in local cohomology is Faltings' Local-global Principle for the Finiteness Dimension of local cohomology modules [8, Satz 1], which states that for a positive integer r , the $R_{\mathfrak{p}}$ -module $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is finitely generated for all $i \leq r$ and for all $\mathfrak{p} \in \text{Spec } R$ if and only if the R -module $H_{\mathfrak{a}}^i(M)$ is finitely generated for all $i \leq r$.

Another formulation of the Faltings' local-global principal, particularly relevant for this paper, is in terms of the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to \mathfrak{a} , where

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\}. \quad (\dagger)$$

With the usual convention that the infimum of the empty set of integers is interpreted as ∞ . We can restate the Faltings' local-global principal (\dagger) , in the following form

$$f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec}(R) \iff f_{\mathfrak{a}}(M) > r. \quad (\dagger\dagger)$$

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Now, let \mathfrak{b} be a second ideal of R . Recall that the \mathfrak{b} -finiteness dimension of M relative to \mathfrak{a} is defined by

$$\begin{aligned} f_{\mathfrak{a}}^{\mathfrak{b}}(M) &:= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b} \not\subseteq \text{Rad}(0 :_R H_{\mathfrak{a}}^i(M))\} \\ &= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b}^n H_{\mathfrak{a}}^i(M) \neq 0 \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

So it is rather natural to ask whether the Faltings' local-global principal, as stated in $(\dagger\dagger)$, generalizes in the obvious way to the invariants $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. In other words, is the statement

$$f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec}(R) \iff f_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \quad (\dagger\dagger\dagger)$$

true for each integer $r > 0$? We shall say that the local-global principle for the annihilation of local cohomology modules holds at level r (over the ring R) if $(\dagger\dagger\dagger)$ is true (for the given r) for every choice of ideals $\mathfrak{a}, \mathfrak{b}$ of R and every choice of finitely generated R -module M .

In fact, recently Brodmann et al. in [5] defined and studied the concept of the local-global principle for annihilation of local cohomology modules at level $r \in \mathbb{N}$ for the ideals \mathfrak{a} and \mathfrak{b} of R . It is shown in [5] that the local-global principle for the annihilation of local cohomology modules holds at levels 1, 2, over an arbitrary commutative Noetherian ring R and at all levels whenever $\dim R \leq 4$.

More recently, for a non-negative integer n , Doustimehr and Naghipour in [7], defined the n th \mathfrak{b} -finiteness dimension of M relative to \mathfrak{a} (resp. the n th \mathfrak{b} -minimum \mathfrak{a} -adjusted depth of M) by

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = \inf\{i \in \mathbb{N}_0 \mid \dim \text{Supp } \mathfrak{b}^t H_{\mathfrak{a}}^i(M) \geq n \text{ for all } t \in \mathbb{N}_0\}.$$

(resp. $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = \inf\{\lambda_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(R), \dim R/\mathfrak{p} \geq n\}$), and using the theory of G-dimension, gave a nice generalization of Faltings' Annihilator theorem.

Now, for a non-negative integer n , we define the upper n th \mathfrak{b} -finiteness dimension of M relative to \mathfrak{a} by

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(R), \dim(R/\mathfrak{p}) \geq n\}.$$

Note that $f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ is either a positive integer or ∞ , and

$$f_{\mathfrak{a}}^{\mathfrak{a}}(M)^n = f_{\mathfrak{a}}^{\mathfrak{a}}(M)_n := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp}(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq n\},$$

is the n th finiteness dimension of M relative to \mathfrak{a} (cf. [2]).

The purpose of the present paper is to show that, if the set $\text{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite or $f_{\mathfrak{a}}^{\mathfrak{b}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$, then the Faltings' local-global principle holds at all levels $r \in \mathbb{N}_0$, where $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ denotes the first non \mathfrak{b} -cofiniteness of local cohomology module $H_{\mathfrak{a}}^i(M)$. As a consequence of this, we provide a short proof of the Faltings' local-global principle for finiteness dimensions. More precisely, as a main result in the second section, we shall show that

Theorem 1.1. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let M be a finitely generated R -module such that the set $\text{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

In particular,

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

The result in Theorem 1.1 is proved in Theorem 2.3 and Corollary 2.4. One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper.

Proposition 1.2. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let M be a finitely generated R -module and n a non-negative integer such that $(\operatorname{Ass}_R H_{\mathfrak{a}}^{f_{\mathfrak{b}}(M)_n}(M))_{\geq n}$ is finite. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n.$$

Here, for any subset T of $\operatorname{Spec}(R)$ and a non-negative integer n , we set

$$(T)_{\geq n} = \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \geq n\}.$$

For a finitely generated R -module M and for ideals $\mathfrak{a}, \mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$, the \mathfrak{b} -cofiniteness dimension $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to \mathfrak{a} is defined by $c_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \text{ is not } \mathfrak{b}\text{-cofinite}\}$. The notion of \mathfrak{b} -cofiniteness dimension $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to \mathfrak{a} is introduced and studied in [3].

In Section 3, it is shown that if $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$, then the local-global principle holds at all levels $r \in \mathbb{N}_0$, for all ideals $\mathfrak{a}, \mathfrak{b}$ of R and each finitely generated R -module M . Moreover, we obtain some new results about the finiteness dimensions of local cohomology modules. In this section among other things, we derive the following consequence of Theorem 1.1, which shows that the Faltings' local-global principle holds at all levels.

Theorem 1.3. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R and let M be a finitely generated R -module such that $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$. Then the Faltings' local-global principle for the finiteness of local cohomology modules holds at all levels.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and \mathfrak{a} will be an ideal of R . Recall that an R -module M is called \mathfrak{a} -cofinite if $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$. The concept of \mathfrak{a} -cofinite modules were introduced by Hartshorne [10]. Also, if n is a non-negative integer, then M is said to be in dimension $< n$, if there is a finitely generated submodule N of M such that $\dim \operatorname{Supp} M/N < n$ (cf. [1, Definition 2.1]); and moreover if T is a subset of $\operatorname{Spec}(R)$, then we define

$$(T)_{\geq n} = \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \geq n\}.$$

Finally, for any ideal \mathfrak{b} of R , the *radical* of \mathfrak{b} , denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [6] and [11].

2. FALTINGS' LOCAL-GLOBAL PRINCIPLE AND FINITENESS

In this section we establish that the Faltings' local-global principle for the finiteness of local cohomology modules holds at all levels over an arbitrary commutative Noetherian ring R , whenever the set $\operatorname{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{b}}(M)}(M))$ is finite. As a consequence, we will provide a new and short proof of the Faltings' local-global principle for finiteness dimensions. We begin with the following lemma which is needed in the proof of Proposition 2.2.

Lemma 2.1. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, and let M be a finitely generated R -module. Then, for every non-negative integer n ,*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n \leq \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)_n.$$

Proof. Let $t := f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ and suppose that $t < f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n$. Then, in view of the definition, there exists an integer m such that $\dim \operatorname{Supp}(\mathfrak{b}^m H_{\mathfrak{a}}^t(M)) < n$. Hence, for every prime ideal \mathfrak{p} of R with $\dim R/\mathfrak{p} \geq n$, we have

$$(\mathfrak{b}^m H_{\mathfrak{a}}^t(M))_{\mathfrak{p}} = 0,$$

and so $(\mathfrak{b} R_{\mathfrak{p}})^m H_{\mathfrak{a} R_{\mathfrak{p}}}^t(M_{\mathfrak{p}}) = 0$, which is a contradiction.

Therefore $f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$. The inequality $f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n \leq \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)_n$, follows easily from the definition and [6, Theorem 9.3.5]. \square

The following proposition plays a key role in the proof of the main result of this section.

Proposition 2.2. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let M be a finitely generated R -module and let n be a non-negative integer such that $(\operatorname{Ass}_R H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n}(M))_{\geq n}$ is finite. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n.$$

Proof. Let $t := f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n$ and suppose that

$$(\operatorname{Ass}_R H_{\mathfrak{a}}^t(M))_{\geq n} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}.$$

In view of Lemma 2.1, it is enough for us to show that $f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n \leq t$. Suppose the contrary that $t < f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$, and look for a contradiction. To this end, in view of definition $t < f_{\mathfrak{a} R_{\mathfrak{p}_i}}^{\mathfrak{b} R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})$, for all $1 \leq i \leq r$. Hence, for all $1 \leq i \leq r$, there exists an integer n_i such that

$$(\mathfrak{b} R_{\mathfrak{p}_i})^{n_i} H_{\mathfrak{a} R_{\mathfrak{p}_i}}^t(M_{\mathfrak{p}_i}) = 0.$$

Set $n := \max\{n_1, \dots, n_r\}$. Then, for all $1 \leq i \leq r$, we have

$$(\mathfrak{b}^n H_{\mathfrak{a}}^t(M))_{\mathfrak{p}_i} = 0.$$

Now, it is easy to see that $\dim \operatorname{Supp}(\mathfrak{b}^n H_{\mathfrak{a}}^t(M)) < n$, which is a contradiction. \square

We are now ready to state and prove the main theorem of this section, which shows that the Faltings' local-global principle for the finiteness of local cohomology modules is valid at all levels over any Noetherian ring R , whenever the set $\operatorname{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite.

Theorem 2.3. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let M be a finitely generated R -module such that the set $\operatorname{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{f_{\mathfrak{a} R_{\mathfrak{p}}}^{\mathfrak{b} R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Proof. Put $n = 0$ in the Proposition 2.2, and use the fact that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}}^{\mathfrak{b}}(M)_0$. \square

An immediate consequence of Theorem 2.3, we give a short proof of the Faltings' local-global principle for finiteness dimensions.

Corollary 2.4. ([6, 9.6.2 Local-global Principle for Finiteness Dimensions]) *Let M be a finitely generated R -module. Then for any ideal \mathfrak{a} of R ,*

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Proof. Put $\mathfrak{a} = \mathfrak{b}$ in Theorem 2.3 and use [4, Corollary 2.3]. \square

3. FALTINGS' LOCAL-GLOBAL PRINCIPLE AND COFINITENESS

The purpose of this section is to show that if $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$, then the local-global principle holds at all levels $r \in \mathbb{N}_0$, for all ideals $\mathfrak{a}, \mathfrak{b}$ of R and each finitely generated R -module M . Moreover, we obtain some new results about the finiteness dimensions of local cohomology modules. In order to do this, let us recall that for a finitely generated R -module M and for ideals $\mathfrak{a}, \mathfrak{b}$ of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, the \mathfrak{b} -cofiniteness dimension $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to \mathfrak{a} is defined by $c_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \text{ is not } \mathfrak{b}\text{-cofinite}\}$. The notion of \mathfrak{b} -cofiniteness dimension $c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to \mathfrak{a} is introduced and studied in [3].

Proposition 3.1. *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and let M be a finitely generated R -module. Then*

$$f_{\mathfrak{a}}(M) = \min\{f_{\mathfrak{a}}^{\mathfrak{b}}(M), c_{\mathfrak{a}}^{\mathfrak{b}}(M)\}.$$

Proof. It is clear that $f_{\mathfrak{a}}(M) \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. Now, if $t := c_{\mathfrak{a}}^{\mathfrak{b}}(M) < f_{\mathfrak{a}}(M)$, then $H_{\mathfrak{a}}^t(M)$ is a finitely generated R -module. Since

$$\operatorname{Supp} H_{\mathfrak{a}}^t(M) \subseteq V(\mathfrak{a}) \subseteq V(\mathfrak{b}),$$

it follows that the R -module $H_{\mathfrak{a}}^t(M)$ is \mathfrak{b} -cofinite, which is a contradiction. Whence $f_{\mathfrak{a}}(M) \leq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$ and so

$$f_{\mathfrak{a}}(M) \leq \min\{f_{\mathfrak{a}}^{\mathfrak{b}}(M), c_{\mathfrak{a}}^{\mathfrak{b}}(M)\}.$$

Finally, suppose that

$$r := f_{\mathfrak{a}}(M) < \min\{f_{\mathfrak{a}}^{\mathfrak{b}}(M), c_{\mathfrak{a}}^{\mathfrak{b}}(M)\},$$

and look for a contradiction. To do this, as $r < f_{\mathfrak{a}}^{\mathfrak{b}}(M)$, there exists an integer n such that $\mathfrak{b}^n H_{\mathfrak{a}}^r(M) = 0$, and thus

$$H_{\mathfrak{a}}^r(M) \cong \operatorname{Hom}_R(R/\mathfrak{b}^n, H_{\mathfrak{a}}^r(M)).$$

Now, since $r < c_{\mathfrak{a}}^{\mathfrak{b}}(M)$, it follows that the R -module $H_{\mathfrak{a}}^r(M)$ is \mathfrak{b} -cofinite, and so it yields from the above isomorphism that $H_{\mathfrak{a}}^r(M)$ is finitely generated, which is a contradiction. \square

We are now ready to state and prove the main theorem of this section, which shows that the Faltings' local-global principle for the finiteness of local cohomology modules is valid at all levels over any Noetherian ring R , whenever $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$.

Theorem 3.2. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R and let M be a finitely generated R -module such that $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$. Then the Faltings' local-global principle for the finiteness of local cohomology modules holds at all levels.*

Proof. Since $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^b(M)$, it follows from Proposition 3.1 that $f_{\mathfrak{a}}^b(M) \leq c_{\mathfrak{a}}^b(M)$, and so $f_{\mathfrak{a}}(M) = f_{\mathfrak{a}}^b(M)$. Now, the assertion follows from Theorem 2.3 and Brodmann-Lashgari's result [4, Corollary 2.3]. \square

Before bringing the next result, we give a couple of lemmas that will be used in the proof of Theorem 3.5.

Lemma 3.3. *Let M be an R -module and let s be a non-negative integer. Then*

(i) *If M is in dimension $< s$, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R/\mathfrak{p} \geq s$.*

(ii) *If M is in dimension $< s$, then the set $(\text{Ass}_R M)_{\geq s}$ is finite.*

Proof. The part (i) follows easily from the definition. In order to prove (ii), since M is in dimension $< s$, it follows from the definition that there is a finitely generated submodule M' of M such that $\dim \text{Supp } M/M' < s$. Now, from the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

we obtain

$$(\text{Ass}_R M)_{\geq s} \subseteq (\text{Ass}_R M')_{\geq s} \cup (\text{Ass}_R M/M')_{\geq s}.$$

As $\dim \text{Supp } M/M' < s$, it follows that $(\text{Ass}_R M/M')_{\geq s} = \emptyset$, and so the set $(\text{Ass}_R M)_{\geq s}$ is finite. \square

Before bringing the next lemma, let us recall that a full subcategory \mathcal{S} of the category of R -modules is called a *Serre subcategory*, when it is closed under taking submodules, quotients and extensions.

Lemma 3.4. *For any non-negative integer n , the class of in dimension $< n$ modules over a Noetherian ring R consists a Serre subcategory of the category of R -modules.*

Proof. See [12, Corollary 2.3]. \square

Theorem 3.5. *Let (R, \mathfrak{m}) be a complete local ring and M a finitely generated R -module. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R and let n be a non-negative integer such that $f_{\mathfrak{a}}^b(M)_n = f_{\mathfrak{a}}^b(M)^n$. Then $f_{\mathfrak{a}}^b(M)_n \geq \text{depth}(\mathfrak{b}, M)$ if and only if $f_{\mathfrak{a}}^n(M) \geq \text{depth}(\mathfrak{b}, M)$.*

Proof. Let $\text{depth}(\mathfrak{b}, M) = s$. Since $\mathfrak{b} \subseteq \mathfrak{a}$, it yields that $f_{\mathfrak{a}}(M) \leq f_{\mathfrak{a}}^b(M)$ and so

$$f_{\mathfrak{a}}^n(M) \leq f_{\mathfrak{a}}^b(M)^n = f_{\mathfrak{a}}^b(M)_n.$$

Recall that $f_{\mathfrak{a}}^n(M) := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp}(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq n\}$.

Now, suppose that $f_{\mathfrak{a}}^b(M)_n \geq s$ and we show that $f_{\mathfrak{a}}^n(M) \geq s$. We use induction on s . For $s = 0$ there is nothing to show. So assume that $s > 0$ and the result has been proved for $s - 1$.

Since $\text{depth}(\mathfrak{b}, M) > 0$, it follows that \mathfrak{b} contains an element x which is a non-zero divisor on M . Moreover, as $f_{\mathfrak{a}}^b(M)_n \geq s$, there exists an integer t such that, for all $i < s$, $\dim \text{Supp}(\mathfrak{b}^t H_{\mathfrak{a}}^i(M)) < n$. Hence from the exact sequence

$$0 \longrightarrow M \xrightarrow{x^t} M \longrightarrow M/x^t M \longrightarrow 0,$$

and [7, Lemma 2.9], there exists an integer l such that, for all $i < s - 1$,

$$\dim \operatorname{Supp}(\mathfrak{b}^l H_{\mathfrak{a}}^i(M/x^t M)) < n,$$

and thus $f_{\mathfrak{a}}^{\mathfrak{b}}(M/x^t M)_n \geq s - 1$. Therefore, in view of the inductive hypothesis

$$f_{\mathfrak{a}}^n(M/x^t M) \geq s - 1,$$

and so by virtue of [1, Theorem 2.5], $H_{\mathfrak{a}}^{s-2}(M/x^t M)$ is in dimension $< n$. Therefore, it follows from the exact sequence and Lemma 3.4

$$0 \longrightarrow H_{\mathfrak{a}}^{s-2}(M)/x^t H_{\mathfrak{a}}^{s-2}(M) \longrightarrow H_{\mathfrak{a}}^{s-2}(M/x^t M) \longrightarrow (0 :_{H_{\mathfrak{a}}^{s-1}(M)} x^t) \longrightarrow 0,$$

that $(0 :_{H_{\mathfrak{a}}^{s-1}(M)} x^t)$ is in dimension $< n$. Hence in view of Lemma 3.3, the $R_{\mathfrak{p}}$ -module $(0 :_{(H_{\mathfrak{a}}^{s-1}(M))_{\mathfrak{p}}} x^t/1)$ is finitely generated, for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$. Now, since $\dim \operatorname{Supp}(\mathfrak{b}^t H_{\mathfrak{a}}^{s-1}(M)) < n$, we have $(\mathfrak{b}^t H_{\mathfrak{a}}^{s-1}(M))_{\mathfrak{p}} = 0$, and hence $x^t/1(H_{\mathfrak{a}}^{s-1}(M))_{\mathfrak{p}} = 0$. Consequently,

$$(H_{\mathfrak{a}}^{s-1}(M))_{\mathfrak{p}} = (0 :_{(H_{\mathfrak{a}}^{s-1}(M))_{\mathfrak{p}}} x^t/1),$$

is a finitely generated $R_{\mathfrak{p}}$ -module. Therefore, by virtue of Lemma 3.3, the R -module $H_{\mathfrak{a}}^{s-1}(M)$ is also in dimension $< n$. Consequently, we deduce from [1, Theorem 2.5] that $f_{\mathfrak{a}}^n(M) \geq s$, as required. \square

Corollary 3.6. *Let (R, \mathfrak{m}) be a complete local ring and M a finitely generated R -module. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R and let n be a non-negative integer such that $\operatorname{depth}(\mathfrak{b}, M) \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n$ and the set $\operatorname{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ is finite. Then*

$$\operatorname{depth}(\mathfrak{b}, M) \leq \min\{f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n, f_{\mathfrak{a}}^n(M)\}.$$

Proof. The result follows from Proposition 2.2 and Theorem 3.5. \square

Theorem 3.7. *Let (R, \mathfrak{m}) be a complete local ring and M a finitely generated R -module. Let \mathfrak{a} be an ideal of R and let s, t be non-negative integers. Then the following statements are equivalent.*

- (i) $H_{\mathfrak{a}}^i(M)$ is in dimension $< s$, for all $i < t$.
- (ii) There exists an integer n such that $\dim \operatorname{Supp}(\mathfrak{a}^n H_{\mathfrak{a}}^i(M)) < s$, for all $i < t$.

Proof. In order to show (i) \implies (ii), let i be an integer such that $i < t$. Then, since $H_{\mathfrak{a}}^i(M)$ is in dimension $< s$, in view of Lemma 3.3, the set

$$(\operatorname{Ass}_R H_{\mathfrak{a}}^i(M))_{\geq s}$$

is finite. Let

$$(\operatorname{Ass}_R H_{\mathfrak{a}}^i(M))_{\geq s} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

Thus, for all j with $1 \leq j \leq n$, the $R_{\mathfrak{p}_j}$ -module $(H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j}$ is finitely generated and thus there exists $n_j \in \mathbb{N}$ such that $(\mathfrak{a}^{n_j} H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$.

Put $n := \max\{n_1, \dots, n_n\}$. Then, for all j with $1 \leq j \leq n$, we have $(\mathfrak{a}^n H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$. Now, we show that $\dim \operatorname{Supp}(\mathfrak{a}^n H_{\mathfrak{a}}^i(M)) < s$. For this, let $\mathfrak{q} \in \operatorname{Ass}_R(\mathfrak{a}^n H_{\mathfrak{a}}^i(M))$. Then,

if $\dim R/\mathfrak{q} \geq s$, then there exists j such that $\mathfrak{q} = \mathfrak{p}_j$ and this is a contradiction. Thus $\dim R/\mathfrak{q} < s$, as desired.

Now, we show (ii) \implies (i). To do this, for all $i < t$ and for all prime ideals \mathfrak{p} with $\dim R/\mathfrak{p} \geq s$, we have $(\mathfrak{a}^n H_{\mathfrak{a}}^i(M))_{\mathfrak{p}} = 0$. Thus, for all $i < t$ and for all prime ideals \mathfrak{p} with $\dim R/\mathfrak{p} \geq s$, the $R_{\mathfrak{p}}$ -module $H_{\mathfrak{a}}^i(M)_{\mathfrak{p}}$ is finitely generated. Hence $f_{\mathfrak{a}}^s(M) \geq t$. Now, since

$$f_{\mathfrak{a}}^s(M) = \inf\{0 \leq i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < s\},$$

(cf. [1, Theorem 2.5]), it follows that $H_{\mathfrak{a}}^i(M)$ is in dimension $< s$ for all $i < t$, as required. \square

Corollary 3.8. *Let (R, \mathfrak{m}) be a complete local ring, M a finitely generated R -module and \mathfrak{a} an ideal of R . Then for any non-negative integer n ,*

$$f_{\mathfrak{a}}^n(M) = f_{\mathfrak{a}}^{\mathfrak{a}}(M)_n.$$

Proof. The assertion follows from Theorem 3.7 and the definitions of $f_{\mathfrak{a}}^n(M)$ and $f_{\mathfrak{a}}^{\mathfrak{a}}(M)_n$. \square

Theorem 3.9. *Let R be a Noetherian ring, M a finitely generated R -module and let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R . Then, for any non-negative integers i and s , the following statements are equivalent.*

- (i) *There exists an integer n such that $\dim \operatorname{Supp}(\mathfrak{b}^n H_{\mathfrak{a}}^i(M)) < s$.*
- (ii) *There exists an integer m such that $\mathfrak{b}^m H_{\mathfrak{a}}^i(M)$ is in dimension $< s$.*

Proof. The implication (i) \implies (ii) is clear. In order to show (ii) \implies (i), since $\mathfrak{b}^m H_{\mathfrak{a}}^i(M)$ is in dimension $< s$, it follows from Lemma 3.3 that the set

$$(\operatorname{Ass}_R \mathfrak{b}^m H_{\mathfrak{a}}^i(M))_{\geq s}$$

is finite. Let

$$(\operatorname{Ass}_R \mathfrak{b}^m H_{\mathfrak{a}}^i(M))_{\geq s} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}.$$

Then, for all j with $1 \leq j \leq r$, the $R_{\mathfrak{p}_j}$ -module $(\mathfrak{b}^m H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j}$ is finitely generated. Hence, there exists $n_j \in \mathbb{N}$ such that $(\mathfrak{a}^{n_j}(\mathfrak{b}^m H_{\mathfrak{a}}^i(M)))_{\mathfrak{p}_j} = 0$. Since $\mathfrak{b} \subseteq \mathfrak{a}$, it follows that $(\mathfrak{b}^{m+n_j} H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$.

Set $n := \max\{m + n_1, \dots, m + n_r\}$. Then, for all j with $1 \leq j \leq r$, we deduce that $(\mathfrak{b}^n H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$.

We show that $\dim \operatorname{Supp}(\mathfrak{b}^n H_{\mathfrak{a}}^i(M)) < s$. To do this, let $\mathfrak{p} \in \operatorname{Ass}_R(\mathfrak{b}^n H_{\mathfrak{a}}^i(M))$. Then $\mathfrak{p} \in \operatorname{Ass}_R(\mathfrak{b}^m H_{\mathfrak{a}}^i(M))$. Now, if $\dim R/\mathfrak{p} \geq s$, then there exists j such that $\mathfrak{p} = \mathfrak{p}_j$ and this is a contradiction. Thus $\dim R/\mathfrak{p} < s$, as required. \square

Corollary 3.10. *Let R be a Noetherian ring and M a finitely generated R -module. Let \mathfrak{a} and \mathfrak{b} be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Then, for any non-negative integer n ,*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = \inf\{0 \leq i \in \mathbb{Z} \mid \mathfrak{b}^m H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < n \text{ for all } m\}.$$

Proof. The assertion follows from Theorem 3.9 and the definition of $f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n$. \square

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